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Lipschitzian Quantum Stochastic Differential Inclusions

G. O. S. Ekhaguere[1,2]

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Quantum stochastic differential inclusions are introduced and studied within the framework of the Hudson–Parthasarathy formulation of quantum stochastic calculus. Results concerning the existence of solutions of a Lipschitzian quantum stochastic differential inclusion and the relationship between the solutions of such an inclusion and those of its convexification are presented. These generalize the Filippov existence theorem and the Filippov–Ważewski relaxation theorem for classical differential inclusions to the present noncommutative setting.

1. INTRODUCTION

There are interesting motivations (Hermes, 1970) for studying classical differential inclusions. For an account of these and a review of some of the important developments in the study of classical differential inclusions up to 1984, see Aubin and Cellina (1984). In this paper, we introduce and study *quantum stochastic differential inclusions* within the framework of the Hudson and Parthasarathy (1984) formulation of quantum stochastic calculus. Such inclusions occur in, for example, quantum stochastic control theory and the theory of quantum stochastic differential equations with discontinuous coefficients. We present results concerning the existence of solutions of a Lipschitzian quantum stochastic differential inclusion and the relationship between the solutions of such an inclusion and those of its convexification.

The rest of the paper is organized as follows. In Section 2, some of the concepts and structures which feature in the subsequent analysis are outlined. The stochastic processes discussed in the sequel are noncommutative; these and the boson stochastic integrators employed in the Hudson and

[1]International Centre for Theoretical Physics, P.O. Box 586, I-34100 Trieste, Italy.
[2]Permanent address: Department of Mathematics, University of Ibadan, Ibadan, Federal Republic of Nigeria.

Parthasarathy (1984) stochastic calculus are introduced in Section 3. Our formulation of quantum stochastic differential inclusions is presented in Section 4. In Section 5, we define Lipschitzian multifunctions and obtain a relationship between such multifunctions and their convexifications. In Section 6, equivalent forms of the stochastic differential inclusions introduced in Section 4 are established. These equivalent forms are inclusions which are, in general, different from classical differential inclusions. Section 7 contains a list of some fundamental statements that are employed in the remaining two sections. The main results of this paper are established in Sections 8 and 9. In Section 8, the existence of solutions to Lipschitzian quantum stochastic differential inclusions is established (Theorem 8.2). This result also represents a generalization of the Gronwall–Filippov inequality (Aubin and Cellina, 1984) to the present noncommutative setting. Finally, Section 9 contains a closure theorem (Theorem 9.1) which establishes a relationship between the solutions of a Lipschitzian quantum stochastic differential inclusion and those of its convexification.

2. FUNDAMENTAL CONCEPTS AND STRUCTURES

We begin by first outlining some of the concepts and structures which are employed in the sequel.

To each pair $(\mathbf{D}, \mathbf{H})$ consisting of a pre-Hilbert space $\mathbf{D}$ and its completion $\mathbf{H}$ we associate the set $L_w^+(\mathbf{D}, \mathbf{H})$ of all linear maps x from $\mathbf{D}$ into $\mathbf{H}$ with the property that the domain of the operator adjoint x^* of x contains $\mathbf{D}$. The members of $L_w^+(\mathbf{D}, \mathbf{H})$ are densely-defined linear operators on $\mathbf{H}$ which do not necessarily leave $\mathbf{D}$ invariant and $L_w^+(\mathbf{D}, \mathbf{H})$ is a linear space when equipped with the usual notions of addition and scalar multiplication. We remark that $L_w^+(\mathbf{D}, \mathbf{H})$ may be additionally endowed with the structure of a *partial *-algebra* in a natural way; for details of this, see Antoine and Mathot (1987).

To $\mathbf{H}$ also corresponds a Hilbert space $\Gamma(\mathbf{H})$, called the *boson Fock space* determined by $\mathbf{H}$. A natural dense subset of $\Gamma(\mathbf{H})$ consists of the linear space generated by the set of *exponential vectors* (Guichardet, 1972) in $\Gamma(\mathbf{H})$ of the form

$$\mathbf{e}(f) = \bigoplus_{n=0}^{\infty} (n!)^{-1/2} \bigotimes^{n} f, \qquad f \in \mathbf{H}$$

where $\otimes^0 f = 1$ and $\otimes^n f$ is the n-fold tensor product of f with itself for $n \geq 1$.

In what follows, $\mathbb{D}$ is some pre-Hilbert space whose completion is $\mathfrak{R}$, and Υ is a fixed Hilbert space.

We write $L^2_\Upsilon(\mathbb{R}_+)$ (resp. $L^2_\Upsilon([0, t))$; resp. $L^2_\Upsilon([t, \infty))$, $t\in\mathbb{R}_+$) for the Hilbert space of square integrable, Υ-valued maps on $\mathbb{R}_+\equiv[0, \infty)$ (resp. on $[0, t)$; resp. on $[t, \infty)$ $t\in\mathbb{R}_+$).

The noncommutative stochastic processes discussed in the sequel are densely-defined linear operators on $\mathfrak{R}\otimes\Gamma(L^2_\Upsilon(\mathbb{R}_+))$; the inner product of this Hilbert space will be denoted by $\langle\cdot, \cdot\rangle$.

For each $t>0$, the direct sum decomposition

$$L^2_\Upsilon(\mathbb{R}_+)=L^2_\Upsilon([0, t))\oplus L^2_\Upsilon([t, \infty))$$

induces a factorization

$$\Gamma(L^2_\Upsilon(\mathbb{R}_+))=\Gamma(L^2_\Upsilon([0, t)))\otimes\Gamma(L^2_\Upsilon([t, \infty)))$$

of Fock space.

Let $\mathbb{E}$, $\mathbb{E}_t$, and $\mathbb{E}^t$, $t>0$, be the linear spaces generated by the exponential vectors in

$$\Gamma(L^2_\Upsilon(\mathbb{R}_+)),\quad \Gamma(L^2_\Upsilon([0, t))),\quad \text{and}\quad \Gamma(L^2_\Upsilon([t, \infty))),\quad t>0$$

respectively. Then, we introduce the definitions

$$\mathscr{A}\equiv L^+_{\mathrm{w}}(\mathbb{D}\underline{\otimes}\mathbb{E}, \mathfrak{R}\otimes\Gamma(L^2_\Upsilon(\mathbb{R}_+)))$$
$$\mathscr{A}_t\equiv L^+_{\mathrm{w}}(\mathbb{D}\underline{\otimes}\mathbb{E}_t, \mathfrak{R}\otimes\Gamma(L^2_\Upsilon([0, t))))\otimes\mathbb{1}^t$$
$$\mathscr{A}^t\equiv\mathbb{1}_t\otimes L^+_{\mathrm{w}}(\mathbb{E}^t, \Gamma(L^2_\Upsilon([t, \infty)))),\qquad t>0$$

where $\underline{\otimes}$ denotes the *algebraic* tensor product and $\mathbb{1}_t$ (resp. $\mathbb{1}^t$) denotes the identity map on $\mathfrak{R}\otimes\Gamma(L^2_\Upsilon([0, t)))$ [resp. $\Gamma(L^2_\Upsilon([t, \infty)))$], $t>0$. It is evident that the spaces $\mathscr{A}_t$ and $\mathscr{A}^t$, $t>0$, may be naturally identified with subspaces of $\mathscr{A}$.

For $\eta, \xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$, define $\|\cdot\|_{\eta,\xi}$ by

$$\|x\|_{\eta,\xi}=|\langle\eta, x\xi\rangle|,\qquad x\in\mathscr{A}$$

Then $\{\|\cdot\|_{\eta,\xi},\ \eta, \xi\in\mathbb{D}\underline{\otimes}\mathbb{E}\}$ is a family of locally convex seminorms on $\mathscr{A}$; we write τ_{w} for the locally convex topology on $\mathscr{A}$ determined by this family.

In the sequel, $\tilde{\mathscr{A}}$, $\tilde{\mathscr{A}}_t$, and $\tilde{\mathscr{A}}^t$ denote the completions of the locally convex spaces $(\mathscr{A}, \tau_{\mathrm{w}})$, $(\mathscr{A}_t, \tau_{\mathrm{w}})$, $(\mathscr{A}^t, \tau_{\mathrm{w}})$, $t>0$, respectively. We remark that the net $\{\tilde{\mathscr{A}}_t : t\in\mathbb{R}_+\}$ is a filtration of $\tilde{\mathscr{A}}$.

Hausdorff topology. If A is a topological space, then clos(A) [resp. comp(A)] denotes the collection of all nonvoid *closed* (resp. *compact*) subsets of A.

We shall employ the *Hausdorff topology* on clos$(\tilde{\mathscr{A}})$. This is defined as follows.

For $x \in \tilde{\mathscr{A}}$, $\mathscr{M}, \mathscr{N} \in \text{clos}(\tilde{\mathscr{A}})$, and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, set

$$\mathbf{d}_{\eta,\xi}(x, \mathscr{N}) \equiv \inf_{y \in \mathscr{N}} \|x - y\|_{\eta,\xi}$$

$$\delta_{\eta,\xi}(\mathscr{M}, \mathscr{N}) \equiv \sup_{x \in \mathscr{M}} \mathbf{d}_{\eta,\xi}(x, \mathscr{N})$$

and

$$\rho_{\eta,\xi}(\mathscr{M}, \mathscr{N}) \equiv \max(\delta_{\eta,\xi}(\mathscr{M}, \mathscr{N}), \delta_{\eta,\xi}(\mathscr{N}, \mathscr{M}))$$

Then $\{\rho_{\eta,\xi}(\cdot): \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ is a family of *pseudometrics* (Kelly; 1955) which determines a Hausdorff topology on $\text{clos}(\tilde{\mathscr{A}})$ denoted in the sequel by τ_{H}.

If $\mathscr{M} \in \text{clos}(\tilde{\mathscr{A}})$, then $\|\mathscr{M}\|_{\eta,\xi}$ is defined by

$$\|\mathscr{M}\|_{\eta,\xi} \equiv \rho_{\eta,\xi}(\mathscr{M}, \{0\})$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

Similarly, for $A, B \in \text{clos}(\mathbb{C})$ and $x \in \mathbb{C}$, the complex numbers, let

$$\mathbf{d}(x, A) \equiv \inf_{y \in A} |x - y|$$

$$\delta(A, B) \equiv \sup_{x \in A} \mathbf{d}(x, B)$$

and

$$\rho(A, B) \equiv \max(\delta(A, B), \delta(B, A))$$

Then, ρ induces a metric topology on $\text{clos}(\mathbb{C})$.

Sets: We employ the usual set-theoretic operations, such as

$$A + B = \{a + b : a \in A \text{ and } b \in B\}$$

$$a + B = \{a + b : b \in B\}$$

for sets A, B and a point a.

3. BOSON STOCHASTIC INTEGRATORS

Let $I \subseteq \mathbb{R}_+$. A *stochastic process* indexed by I is an $\tilde{\mathscr{A}}$-valued map on I.

A stochastic process X is called *adapted* if $X(t) \in \tilde{\mathscr{A}}_t$ for each $t \in I$.

We write $\text{Ad}(\tilde{\mathscr{A}})$ for the set of all adapted stochastic processes indexed by I.

Definition. A member X of $\text{Ad}(\tilde{\mathscr{A}})$ is called (i) *weakly absolutely continuous* if the map $t \mapsto \langle \eta, X(t)\xi \rangle$, $t \in I$, is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$; (ii) *locally absolutely p-integrable* if $\|X(\cdot)\|^p_{\eta,\xi}$ is Lebesgue-measurable and integrable on $[0, t) \subseteq I$ for each $t \in I$ and arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

Notation. We write $\mathrm{Ad}(\tilde{\mathscr{A}})_{\mathrm{wac}}$ [resp. $L^p_{\mathrm{loc}}(\tilde{\mathscr{A}})$] for the set of all *weakly absolutely continuous* (resp. *locally absolutely p-integrable*) members of $\mathrm{Ad}(\tilde{\mathscr{A}})$.

Stochastic integrators. Let $L^\infty_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$ [resp. $L^\infty_{B(\Upsilon),\mathrm{loc}}(\mathbb{R}_+)$] be the linear space of all measurable, locally bounded functions from $\mathbb{R}_+$ to Υ [resp. to $B(\Upsilon)$, the Banach space of bounded endomorphisms of Υ]. If $f\in L^\infty_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$ and $\pi\in L^\infty_{B(\Upsilon),\mathrm{loc}}(\mathbb{R}_+)$, then πf is the member of $L^\infty_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$ given by $(\pi f)(t)=\pi(t)f(t)$, $t\in\mathbb{R}_+$.

For $f\in L^2_\Upsilon(\mathbb{R}_+)$ and $\pi\in L^\infty_{B(\Upsilon),\mathrm{loc}}(\mathbb{R}_+)$, define the operators $a(f)$, $a^+(f)$, and $\lambda(\pi)$ in $L^+_w(\mathbb{D}, \Gamma(L^2_\Upsilon(\mathbb{R}_+)))$ as follows:

$$a(f)\mathrm{e}(g)=\langle f,g\rangle_{L^2_\Upsilon(\mathbb{R}_+)}\mathrm{e}(g)$$

$$a^+(f)\mathrm{e}(g)=\frac{d}{d\sigma}\,\mathrm{e}(g+\sigma f)|_{\sigma=0}$$

$$\lambda(\pi)\mathrm{e}(g)=\frac{d}{d\sigma}\,\mathrm{e}(e^{\sigma\pi}f)|_{\sigma=0}$$

$g\in L^2_\Upsilon(\mathbb{R}_+)$.

These are the *annihilation*, *creation*, and *gauge* operators of quantum field theory. For arbitrary $f\in L^\infty_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$ and $\pi\in L^\infty_{B(\Upsilon),\mathrm{loc}}(\mathbb{R}_+)$, they give rise to the operator-valued maps A_f, A_f^+, and Λ_π defined by

$$A_f(t)\equiv a(f\chi_{[0,t)})$$

$$A_f^+(t)\equiv a^+(f\chi_{[0,t)})$$

$$\Lambda_\pi(t)\equiv\lambda(\pi\chi_{[0,t)})$$

$t\in\mathbb{R}_+$, where χ_I denotes the indicator function of the Borel set $I\subseteq\mathbb{R}_+$.

The maps A_f, A_f^+, and Λ_π are stochastic processes, called the *annihilation, creation,* and *gauge* processes, respectively, when their values are identified (as we do henceforth) with their amplifications on $\mathfrak{R}\otimes\Gamma(L^2_\Upsilon(\mathbb{R}_+))$. These are the *stochastic integrators* in the Hudson and Parthasarathy (1984) formulation of boson quantum stochastic integration, which we adopt in the sequel.

Accordingly, if $p, q, u, v\in L^2_{\mathrm{loc}}(\tilde{\mathscr{A}})$, $f, g\in L^\infty_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$, and

$$\pi\in L^\infty_{B(\Upsilon),\mathrm{loc}}(\mathbb{R}_+)$$

then we interpret the integral

$$\int_{t_0}^{t} p(s)\,d\Lambda_\pi(s)+q(s)\,dA_f(s)+u(s)\,dA_g^+(s)+v(s)\,ds,\qquad t_0,t\in\mathbb{R}_+\quad(3.1)$$

as in Hudson and Parthasarathy (1984).

4. STOCHASTIC DIFFERENTIAL INCLUSIONS

The rest of this paper will deal mainly with stochastic differential inclusions involving multivalued stochastic processes.

Definition. 1. By a *multivalued stochastic process* indexed by $I \subseteq \mathbb{R}_+$ we mean a multifunction on I with values in $\operatorname{clos}(\tilde{\mathscr{A}})$.

2. If Φ is a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, then a *selection* of Φ is a stochastic process $X: I \to \tilde{\mathscr{A}}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$.

3. A multivalued stochastic process Φ will be called (i) *adapted* if $\Phi(t) \subseteq \tilde{\mathscr{A}}_t$ for each $t \in \mathbb{R}_+$; (ii) *measurable* if $t \mapsto d_{\eta,\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \tilde{\mathscr{A}}$, $\eta, \xi \in \mathbb{D} \otimes \mathbb{E}$; (iii) *locally absolutely p-integrable* if $t \mapsto \|\Phi(t)\|_{\eta,\xi}$, $t \in \mathbb{R}_+$, lies in $L^p_{\text{loc}}(I)$ for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

Notation. 1. The set of all locally absolutely p-integrable multivalued stochastic processes will be denoted by $L^p_{\text{loc}}(\tilde{\mathscr{A}})_{\text{mvs}}$.

2. For $p \in (0, \infty)$ and $I \subseteq \mathbb{R}_+$, $L^p_{\text{loc}}(I \times \tilde{\mathscr{A}})_{\text{mvs}}$ is the set of maps $\Phi: I \times \tilde{\mathscr{A}} \to \operatorname{clos}(\tilde{\mathscr{A}})$ such that $t \mapsto \Phi(t, X(t))$, $t \in I$, lies in $L^p_{\text{loc}}(\tilde{\mathscr{A}})_{\text{mvs}}$ for every $X \in L^p_{\text{loc}}(\tilde{\mathscr{A}})$.

3. If $\Phi \in L^p_{\text{loc}}(I \times \tilde{\mathscr{A}})_{\text{mvs}}$, then

$$L_p(\Phi) \equiv \{\varphi \in L^p_{\text{loc}}(\tilde{\mathscr{A}}): \varphi \text{ is a selection of } \Phi\}$$

4. In the sequel, $f, g \in L^\infty_{\Upsilon,\text{loc}}(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\Upsilon),\text{loc}}(\mathbb{R}_+)$, $\mathbb{1}$ is the identity map on $\mathfrak{R} \otimes \Gamma(L^2_\Upsilon(\mathbb{R}_+))$, and M is any of the stochastic processes A_f, A_g^+, Λ_π, and $s \mapsto s\mathbb{1}$, $s \in \mathbb{R}_+$.

We introduce *stochastic integral* (resp. *differential*) *expressions* as follows.

If $\Phi \in L^2_{\text{loc}}(I \times \tilde{\mathscr{A}})_{\text{mvs}}$ and $(t, X) \in I \times L^2_{\text{loc}}(\tilde{\mathscr{A}})$, then we make the definition

$$\int_{t_0}^t \Phi(s, X(s))\, dM(s) \equiv \left\{ \int_{t_0}^t \varphi(s)\, dM(s): \varphi \in L_2(\Phi) \right\}$$

This leads to the following notion.

Definition. Let $E, F, G, H \in L^2_{\text{loc}}(I \times \tilde{\mathscr{A}})_{\text{mvs}}$ and (t_0, x_0) be a fixed point of $I \times \tilde{\mathscr{A}}$. Then, a relation of the form

$$X(t) \in x_0 + \int_{t_0}^t (E(s, X(s))\, d\Lambda_\pi + F(s, X(s))\, dA_f(s) + G(s, X(s))\, dA_g^+(s) + H(s, X(s))\, ds), \qquad t \in I$$

will be called a *stochastic integral inclusion* with *coefficients E, F, G,* and *H* and *initial data* (t_0, x_0). As an abbreviation, we shall sometimes write the foregoing inclusion as follows:

$$dX(t) \in E(t, X(t))\, d\Lambda_\pi(t) + F(t, X(t))\, dA_f(t) + G(t, X(t))\, dA_g^+(t) + H(t, X(t))\, dt \tag{4.1a}$$

almost all $t \in I$,

$$X(t_0) = x_0 \tag{4.1b}$$

and refer to this as a *stochastic differential inclusion* with *coefficients E, F, G,* and *H* and *initial data* (t_0, x_0).

Definition. By a *solution* of (4.1), we mean a *weakly absolutely continuous* stochastic process $\varphi \in L^2_{\text{loc}}(\tilde{\mathscr{A}})$ such that

$$d\varphi(t) \in E(t, \varphi(t))\, d\Lambda_\pi(t) + F(t, \varphi(t))\, dA_f(t) + G(t, \varphi(t))\, dA_g^+(t) + H(t, \varphi(t))\, dt$$

almost all $t \in I$,

$$\varphi(t_0) = x_0$$

Remark. 1. We shall prove the existence of solutions to a stochastic differential inclusion with Lipschitzian coefficients.

2. If $\mathscr{M}$ is a subset of $\tilde{\mathscr{A}}$, we write co $\mathscr{M}$ for the *closed convex hull* of $\mathscr{M}$ and if $\Phi: I \times \tilde{\mathscr{A}} \to \text{clos}(\tilde{\mathscr{A}})$, we define co $\Phi: I \times \tilde{\mathscr{A}} \to \text{clos}(\tilde{\mathscr{A}})$ by

$$(\text{co}\, \Phi)(t, x) = \text{co}\, \Phi(t, x), \qquad (t, x) \in I \times \tilde{\mathscr{A}}$$

3. Related to (4.1) is the following stochastic differential inclusion:

$$dX(t) \in \text{co}\, E(t, X(t))\, d\Lambda_\pi(t) + \text{co}\, F(t, X(t))\, dA_f(t) + \text{co}\, G(t, X(t))\, dA_g^+(t) + \text{co}\, H(t, X(t))\, dt \tag{4.2a}$$

almost all $t \in I$,

$$X(t_0) = x_0 \tag{4.2b}$$

4. Equivalent forms of (4.1) and (4.2) are established in Section 6 and the relationship between the solutions of (4.1) and (4.2) is investigated in Section 9.

5. Two instances where quantum stochastic differential inclusions arise are the following.

Quantum stochastic control theory. In this theory, there is a preassigned compact subset $\mathscr{C}$, called the space of admissible controls, of some topological space and one searches for solutions of the stochastic differential equation:

$$\begin{aligned} dX(t) &= p(t, X(t), c(t))\, d\Lambda_\pi(t) + q(t, X(t), c(t))\, dA_f(t) \\ &\quad + u(t, X(t), c(t))\, dA_g^+(t) + q(t, X(t), c(t))\, dt \qquad (4.3) \\ X(t_0) &= x_0 \end{aligned}$$

for almost all $t\in I$, where $c: I \to \mathscr{C}$ and p, q, u, and v are maps from $I\times\tilde{\mathscr{A}}\times\mathscr{C}\to\tilde{\mathscr{A}}$ such that $p(\cdot, Z(\cdot), c(\cdot))$, $q(\cdot, Z(\cdot), c(\cdot))$, $u(\cdot, Z(\cdot), c(\cdot))$, and $v(\cdot, Z(\cdot), c(\cdot))$ are adapted and lie in $L^2_{loc}(\tilde{\mathscr{A}})$ for all $Z\in L^2_{loc}(\tilde{\mathscr{A}})$ and $c\in\mathscr{C}$. If we introduce the multivalued stochastic processes E, F, G, H defined by

$$\begin{aligned} E(t, x) &\equiv \{p(t, x, c): c\in\mathscr{C}\} \\ F(t, x) &\equiv \{q(t, x, c): c\in\mathscr{C}\} \\ G(t, x) &\equiv \{u(t, x, c): c\in\mathscr{C}\} \\ H(t, x) &\equiv \{v(t, x, c): c\in\mathscr{C}\} \end{aligned}$$

$(t, x)\in I\times\tilde{\mathscr{A}}$, then (4.3) may be written formally as (4.1) and the problem of the existence of solutions to (4.3) for all admissible controls is reduced to the problem of the existence of solutions to (4.1) with these definitions of E, F, G, H.

Regularization of stochastic differential equations. In this case, one encounters stochastic differential equations of the form

$$\begin{aligned} dX(t) &= p(X(t))\, d\Lambda_\pi(t) + q(X(t))\, dA_f(t) + u(X(t))\, dA_g^+(t) \\ &\quad + q(X(t))\, dt \qquad (4.4) \\ X(t_0) &= x_0 \end{aligned}$$

for almost all $t\in I$, where p, q, u, v are *discontinuous* maps from $\tilde{\mathscr{A}}\to\tilde{\mathscr{A}}$ such that if $Z\in L^2_{loc}(\tilde{\mathscr{A}})$, and $p(Z(t))$, $q(Z(t))$, $u(Z(t))$, and $v(Z(t))$ are defined for almost all $t\in I$, then the maps $p(Z(\cdot))$, $q(Z\cdot))$, $u(Z(\cdot))$, and $v(Z(\cdot))$ are adapted and lie in $L^2_{loc}(\tilde{\mathscr{A}})$. Such a stochastic differential equation is said to be *discontinuous*. To discuss the problem of the existence of solutions to this equation, one may introduce the minimal upper semicontinuous multifunctions E, F, G, and H on $\tilde{\mathscr{A}}$ with convex values in $\text{clos}(\tilde{\mathscr{A}})$ whose graphs (Aubin and Cellina, 1984) contain the graphs of p, q, u, and v, respectively. Then, $E(x)=\{p(x)\}$, $F(x)=\{q(x)\}$, $G(x)=\{u(x)\}$, and $H(x)=\{v(x)\}$ at each point x of continuity of p, q, u, and v and one gets that any solution to (4.4) is a solution to the inclusion

$$dX(t) \in E(X(t))\, d\Lambda_\pi(t) + F(X(t))\, dA_f(t) + G(X(t))\, dA_g^+(t) + H(X(t))\, dt \tag{4.5}$$

$$X(t_0) = x_0$$

for almost all $t \in I$. Moreover, if φ is a solution of (4.5) and p, q, u, and v are continuous at $\varphi(t)$ for almost all $t \in I$, then

$$d\varphi(t) = p(\varphi(t))\, d\Lambda_\pi(t) + q(\varphi(t))\, dA_f(t) + u(\varphi(t))\, dA_g^+(t) + v(\varphi(t))\, dt$$

$$X(t_0) = x_0$$

for almost all $t \in I$, i.e., φ is a solution of (4.4).

5. LIPSCHITZIAN MULTIFUNCTIONS

These are explained as follows.

Let $\mathcal{N} \in \text{clos}(\tilde{\mathcal{A}})$ and $I \subseteq \mathbb{R}_+$.

Definition. A map $\Phi: I \times \mathcal{N} \to \text{clos}(\tilde{\mathcal{A}})$ will be called *Lipschitzian* if for each $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, there exists $k^{\Phi}_{\eta,\xi}: I \to (0, \infty)$ in $L^1_{\text{loc}}(I)$ such that

$$\rho_{\eta,\xi}(\Phi(t, x), \Phi(t, x)) \leq k^{\Phi}_{\eta,\xi}(t) \|x - y\|_{\eta,\xi}$$

for all $x, y \in \mathcal{N}$ and almost all $t \in I$.

The functions $\{k^{\Phi}_{\eta,\xi}(\cdot): \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ will be called *Lipschitz functions* for Φ; these are constants if Φ does not depend explicitly on t.

Remark. 1. The notion of a Lipschitzian map from $I \times \mathcal{N}$ to $\text{clos}(\mathbb{C})$ is analogously formulated using the Hausdorff metric ρ.

2. In the sequel, if Φ is a map from $I \times \mathcal{N}$ into the set of multivalued sesquilinear forms on $\mathbb{D} \underline{\otimes} \mathbb{E}$, i.e., if Φ maps $I \times \mathcal{N}$ to $2^{\text{sesq}(\mathbb{D} \underline{\otimes} \mathbb{E})}$, where $\text{sesq}(\mathbb{D} \underline{\otimes} \mathbb{E})$ is the linear space of sesquilinear forms on $\mathbb{D} \underline{\otimes} \mathbb{E}$, then for $(t, x) \in I \times \mathcal{N}$, the value of $\Phi(t, x)$ at $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ will be denoted by $\Phi(t, x)(\eta, \xi)$. Such a map Φ will be called Lipschitzian (resp. continuous) if for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ the map $(t, x) \mapsto \Phi(t, x)(\eta, \xi)$ from $I \times \mathcal{N}$ to $2^{\mathbb{C}}$ is Lipschitzian (resp. continuous).

3. We shall sometimes employ the following result.

Proposition 5.1. Let

$$\Phi: \tilde{\mathcal{A}} \to \text{clos}(\tilde{\mathcal{A}})$$

[resp. $Q \to \mathrm{clos}(\mathbb{C})$] be a Lipschitzian map with Lipschitz constants

$$\{k^{\Phi}_{\eta,\xi} : \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$$

[resp. $\{k^{Q}_{\eta,\xi} : \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$]. Then, the map co Φ (resp. co Q) is Lipschitzian with the same Lipschitz constants.

Proof. Let Φ be Lipschitzian. Let $\varepsilon > 0$ be arbitrary and suppose $t \in I$ and $x, x' \in \tilde{\mathscr{A}}$. Then, for $y \in \mathrm{co}\,\Phi(x)$ there exist $\{y_i\}$ such that

$$\left\| y - \sum_i \lambda_i y_i \right\|_{\eta,\xi} < \varepsilon, \qquad \forall \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

Next, for each i, there exists $y'_i \in \Phi(x)$ such that

$$\|y - y'_i\|_{\eta,\xi} \le \rho_{\eta,\xi}(\Phi(x), \Phi(x')) + \varepsilon$$

It follows that

$$\begin{aligned} \left\| y - \sum_i \lambda_i y'_i \right\|_{\eta,\xi} &\le \varepsilon + \left\| \sum_i \lambda_i (y_i - y'_i) \right\|_{\eta,\xi} \\ &\le \varepsilon + \sum_i \lambda_i (\rho_{\eta,\xi}(\Phi(x), \Phi(x')) + \varepsilon) \\ &\le 2\varepsilon + k^{\Phi}_{\eta,\varepsilon} \|x - x'\|_{\eta,\xi} \end{aligned}$$

The result now follows by interchanging the roles of x and x'. A similar proof holds for co Q. This concludes the proof. ■

6. EQUIVALENT FORMS OF (4.1) AND (4.2)

Unless otherwise stated, E, F, G, and $H \in L^2_{\mathrm{loc}}(I \times \tilde{\mathscr{A}})_{\mathrm{mvs}}$ and (t_0, x_0) is some fixed point of $I \times \tilde{\mathscr{A}}$.

To establish equivalent forms of (4.1) and (4.2), we take Theorems 4.1 and 4.4 of Hudson and Parthasarthy (1984), which describe the matrix elements of the quantum stochastic integral (3.1), as our point of departure.

For $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = c \otimes \mathbf{e}(\alpha)$ and $\xi = d \otimes \mathbf{e}(\beta)$, define $\mu_{\alpha\beta}$, ν_β, $\sigma_\alpha : I \to \mathbb{C}$ by

$$\mu_{\alpha\beta}(t) = \langle \alpha(t), \pi(t)\beta(t) \rangle_{\Upsilon}$$

$$\nu_\beta(t) = \langle f(t), \beta(t) \rangle_{\Upsilon}$$

$$\sigma_\alpha(t) = \langle \alpha(t), g(t) \rangle_{\Upsilon}$$

$t \in I$. To these functions, we associate the maps μE, νF, σG, P, and co P from $I \times \tilde{\mathscr{A}}$ into the set of multivalued sesquilinear forms on $\mathbb{D} \underline{\otimes} \mathbb{E}$

defined by

$$\begin{aligned}(\mu E)(t, x)(\eta, \xi) &= \{\langle \eta, \mu_{\alpha\beta}(t) p(t, x)\xi\rangle : p(t, x) \in E(t, x)\} \\ (\nu F)(t, x)(\eta, \xi) &= \{\langle \eta, \nu_{\beta}(t) q(t, x)\xi\rangle : q(t, x) \varepsilon F(t, x)\} \\ (\sigma G)(t, x)(\eta, \xi) &= \{\langle \eta, \sigma_{\alpha}(t) u(t, x)\xi\rangle : u(t, x) \in F(t, x)\} \\ P(t, x)(\eta, \xi) &= (\mu E)(t, x)(\eta, \xi) + (\nu F)(t, x)(\eta, \xi) \\ &\quad + (\sigma G)(t, x)(\eta, \xi) + H(t, x)(\eta, \xi) \\ (\mathrm{co}\, P)(t, x)(\eta, \xi) &= \text{closed convex hull of } P(t, x)(\eta, \xi)\end{aligned} \tag{6.1}$$

$\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $(t, x) \in I \times \tilde{\mathscr{A}}$, where

$$H(t, x)(\eta, \xi) = \{v(t, x)(\eta, \xi) : v(\cdot, X(\cdot)) \text{ is a selection of } H(\cdot, X(\cdot)) \ \forall X \in L^2_{\mathrm{loc}}(\tilde{\mathscr{A}})\}$$

$\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $(t, x) \varepsilon I \times \tilde{\mathscr{A}}$.

The map P will play a fundamental role in the subsequent discussion. The following properties of P will be employed.

Proposition 6.1. Let E, F, G, H lie in $L^2_{\mathrm{loc}}(I \times \tilde{\mathscr{A}})_{\mathrm{mvs}}$. Then:

(i) For arbitrary $\eta, \xi \in \mathbb{D} \otimes \mathbb{E}$ and $X \in L^2_{\mathrm{loc}}(\tilde{\mathscr{A}})$, every selection of the map $t \mapsto P(t, X(t))(\eta, \xi)$ lies in $\bar{L}^1_{\mathrm{loc}}(I)$.

(ii) The map P is (a) Lipschitzian whenever E, F, G, H are Lipschitzian; (b) continuous whenever μE, νF, σG, H are continuous.

Proof. (i) This follows from Hudson and Parthasarathy (1984), Theorems 4.1 and 4.4.

(ii) Let $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = c \otimes \mathbf{e}(\alpha)$ and $\xi = d \otimes \mathbf{e}(\beta)$.

(a) This is a consequence of the following, easily derived inequality:

$$\begin{aligned}&\rho(P(t, x)(\eta, \xi), P(t, y)(\eta, \xi)) \\ &\quad \leq |\mu_{\alpha\beta}(t)| \rho_{\eta,\xi}(E(t, x), E(t, y)) \\ &\qquad + |\nu_{\beta}(t)| \rho_{\eta,\xi}(F(t, x), F(t, y)) \\ &\qquad + |\sigma_{\alpha}(t)| \rho_{\eta,\xi}(G(t, x), G(t, y)) + \rho_{\eta,\xi}(H(t, x), H(t, y))\end{aligned}$$

(b) This is a consequence of the following, easily derived inequality:

$$\begin{aligned}&\rho(P(t, x)(\eta, \xi), P(s, y)(\eta, \xi)) \\ &\quad \leq \rho((\mu E)(t, x)(\eta, \xi), (\mu E)(s, y)(\eta, \xi)) \\ &\qquad + \rho((\nu F)(t, x)(\eta, \xi), (\nu F)(s, y)(\eta, \xi)) \\ &\qquad + \rho((\sigma G)(t, x)(\eta, \xi), (\sigma G)(s, y)(\eta, \xi)) \\ &\qquad + \rho(H(t, x)(\eta, \xi), H(s, y)(\eta, \xi))\end{aligned}$$

This concludes the proof. ■

Remark. 1. If P is Lipschitzian, then its Lipschitz functions will be denoted by $\{k^P_{\eta,\xi}(\cdot): \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$.

2. The next result gives the equivalent forms of (4.1) and (4.2) alluded to above.

Theorem 6.2. Let $E, F, G, H \in L^2_{\mathrm{loc}}(I \times \tilde{\mathscr{A}})_{\mathrm{mvs}}$. Then:

(i) Problem (4.1) is equivalent to the following:

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in P(t, X(t))(\eta, \xi), \qquad X(t_0) = x_0 \qquad (6.2\mathrm{a})_P$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, almost all $t \in I$.

(ii) Problem (4.2) is equivalent to the following:

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in (\mathrm{co}\, P)(t, X(t))(\eta, \xi), \qquad X(t_0) = x_0 \qquad (6.2\mathrm{b})_P$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, almost all $t \in I$.

Proof. (i) If (4.2) holds, then there are p, q, u, and v in $L_2(E(\cdot, X(\cdot)))$, $L_2(F(\cdot, X(\cdot)))$, $L_2(G(\cdot, X(\cdot)))$, and $L_2(H(\cdot, X(\cdot)))$, respectively, such that (3.1) holds. It then follows from Hudson and Parthasarathy (1984), Theorems 4.1 and 4.4, that $(6.2\mathrm{a})_P$ holds. Conversely, given $(6.2\mathrm{b})_P$, there exists $Z(t)(\eta, \xi)$ in $P(t, X(t))(\eta, \xi)$ such that

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle = Z(t)(\eta, \xi)$$

$\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. As $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ are arbitrary, $Z(t)(\eta, \xi)$ has the form

$$Z(t)(\eta, \xi) = \langle \eta, Z(t)\xi\rangle$$

where $Z(t)$ has a representation of the type (3.1) for some p, q, u, and v in $L_2(E(\cdot, X(\cdot)))$, $L_2(F(\cdot, X(\cdot)))$, $L_2(G(\cdot, X(\cdot)))$, and $L_2(H(\cdot, X(\cdot)))$, respectively. Hence, (4.1) holds.

(ii) By (i), it follows that (4.2) is equivalent to an inclusion of the form $(6.2\mathrm{a})_P$, with E, F, G, and H replaced with $\mathrm{co}\, E$, $\mathrm{co}\, F$, $\mathrm{co}\, G$, and $\mathrm{co}\, H$, respectively. Denoting the right-hand side of $(6.2\mathrm{a})_P$ with these replacements by $P^{\mathrm{co}}(t, X(t))(\eta, \xi)$, we need only show that

$$P^{\mathrm{co}}(t, X(t))(\eta, \xi) = (\mathrm{co}\, P)(t, X(t))(\eta, \xi)$$

$\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Since for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ the map $x \mapsto \langle \eta, x\xi\rangle$, is linear and continuous from $\tilde{\mathscr{A}}$ to $\mathbb{C}$, and $P^{\mathrm{co}}(t, X(t))(\eta, \xi)$ is the image of the right-hand side of (4.2) under this map, it follows that $P^{\mathrm{co}}(t, X(t))(\eta, \xi)$ is closed and convex, and coincides with $(\text{co}\, P)(t, X(t))(\eta, \xi)$ for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. This concludes the proof. ■

Remark. Notice that since, in general,

$$P(t, x)(\eta, \xi) \neq \tilde{P}(t, \langle \eta, x\xi \rangle)$$

and

$$(\text{co}\, P)(t, x)(\eta, \xi) \neq (\text{co}\, \tilde{P})(t, \langle \eta, x\xi \rangle)$$

$\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $(t, x) \in I \times \tilde{\mathscr{A}}$, where $\tilde{P}$ is some clos($\mathbb{C}$)-valued map on $I \times \mathbb{C}$, it follows that the inclusions $(6.2\text{a})_P$ and $(6.2\text{b})_P$ are, in general, not of the classical types described in Aubin and Cellina (1984).

7. SOME FUNDAMENTAL STATEMENTS

In connection with the subsequent results, we list the following statements.

(S_1) $Y: I \to \tilde{\mathscr{A}}$ is a weakly absolutely continuous adapted stochastic process with the property that for each $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and almost all $t \in I$, there is a positive number $p_{\eta,\xi}(t)$ such that

$$\mathbf{d}\left(\frac{d}{dt} \langle \eta, Y(t)\xi \rangle, P(t, Y(t))(\eta, \xi)\right) \leq p_{\eta,\xi}(t)$$

(S_2) $\gamma > 0$ is an arbitrary but fixed number and $Q_{Y,\gamma}$ is the set

$$Q_{Y,\gamma} \equiv \{(t, x) \in I \times \tilde{\mathscr{A}} : \|x - Y(t)\|_{\eta,\xi} \leq \gamma \ \forall \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$$

(S_3) Each of the maps E, F, G, H is Lipschitzian from $Q_{Y,\gamma}$ to $(\text{clos}(\tilde{\mathscr{A}}), \tau_H)$.

(S_4) For each $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$,

$$\delta_{\eta,\xi} \equiv \|x_0 - Y(t_0)\|_{\eta,\xi}$$

(S_5) Suppose

$$\delta_{\eta,\xi} \leq \gamma, \qquad \forall \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

(S_6) For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $t \in I$,

$$\Xi_{\eta,\xi}(t) \equiv \delta_{\eta,\xi} \exp\left[\int_{t_0}^{t} ds\, k^{P}_{\eta,\xi}(s)\right] + \int_{t_0}^{t} ds\, p_{\eta,\xi}(s) \exp\left[\int_{s}^{t} dr\, k^{P}_{\eta,\xi}(r)\right]$$

(S_7) J is the subset of I defined by

$$J \equiv \{t \in I : \Xi_{\eta,\xi}(t) \leq \gamma \ \forall \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$$

8. EXISTENCE OF SOLUTIONS TO (4.1)

We consider the problem of the existence of solutions to (4.1).

In the sequel,

$$m_{\eta,\xi}(t) = \int_{t_0}^{t} ds\, k^{P}_{\eta,\xi}(s), \qquad \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, \quad t \in I$$

where $\{k^{P}_{\eta,\xi} : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$ denotes the set of Lipschitz functions for P.

Proposition 8.1. Let $\{\Phi_i\}_{i=1}^{\infty}$ be a sequence of weakly absolutely continuous maps from I to $\tilde{\mathscr{A}}$ which satisfy:

(i) $(t, \Phi_i(t)) \in Q_{Y,\gamma}$, $i \geq 1$, for almost all $t \in J$.

(ii) There exists a sequence $\{V_i\}_{i=0}^{\infty} \subset L^1_{\text{loc}}(\tilde{\mathscr{A}})$ such that (a)

$$\Phi_i(t) = x_0 + \int_{t_0}^{t} ds\, V_{i-1}(s), \qquad i \geq 1$$

and (b)

$$\left|\frac{d}{dt}\langle \eta, \Phi_i(t)\xi\rangle - \langle \eta, \Phi_{i-1}(t)\xi\rangle\right|$$

$$\leq k^{P}_{\eta,\xi}(t)\left\{\frac{\delta_{\eta,\xi}(m_{\eta,\xi}(t))^{i-2}}{(i-2)!} + \int_{t_0}^{t} ds\, \frac{[m_{\eta,\xi}(t) - m_{\eta,\xi}(s)]^{i-2}}{(i-2)!}\, p_{\eta,\xi}(s)\right\}$$

$$\equiv b_{\eta,\xi,i-2}(t)$$

for almost all $t \in J$. Then (c)

$$\|\Phi_i(t) - \Phi_{i-1}(t)\|_{\eta,\xi} \leq b_{\eta,\xi,i-1}(t), \qquad i \geq 2, \quad t \in J$$

Proof. Let (i) and (ii) hold. Then

$$\|\Phi_i(t) - \Phi_{i-1}(t)\|_{\eta,\xi}$$

$$= \left|\int_{t_0}^{t} ds\, \langle \eta, [V_{i-1}(s) - V_{i-2}(s)]\xi\rangle\right| \qquad \text{[by (ii)(a)]}$$

$$=\left|\int_{t_0}^{t} ds \left\{\frac{d}{ds}\langle\eta, \Phi_i(s)\xi\rangle - \frac{d}{ds}\langle\eta, \Phi_{i-1}(s)\xi\rangle\right\}\right| \qquad \text{[by (ii)(a)]}$$

$$\leq \int_{t_0}^{t} ds \left|\frac{d}{ds}\langle\eta, \Phi_i(s)\xi\rangle - \frac{d}{ds}\langle\eta, \Phi_{i-1}(s)\xi\rangle\right|$$

$$\leq \int_{t_0}^{t} ds\, b_{\eta,\xi,i-2}(s) \qquad \text{[by (ii)(b)]}$$

$$= b_{\eta,\xi,i-1}(t), \qquad i \geq 2, \quad t \in J$$

This concludes the proof. ■

Remark. The following is a result concerning the existence of solutions to (4.1). It also furnishes a generalization of the classical Gronwall inequality (Walter, 1964).

Theorem 8.2. Suppose that (S_1)–(S_7) hold and E, F, G, and H are continuous from $I \times \tilde{\mathscr{A}}$ to $(\text{clos}(\tilde{\mathscr{A}}), \tau_H)$. Then, there exists a solution Φ of (4.1) such that

$$\|\Phi(t) - Y(t)\|_{\eta,\xi} \leq \Xi_{\eta,\xi}(t), \qquad t \in J \tag{8.1a}$$

and

$$\left|\frac{d}{dt}\langle\eta, \Phi(t)\xi\rangle - \frac{d}{dt}\langle\eta, Y(t)\xi\rangle\right| \leq k^{P}_{\eta,\xi}(t)\Xi_{\eta,\xi}(t) + p_{\eta,\xi}(t) \tag{8.1b}$$

for almost all $t \in J$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Proof. The proof (as well as that of Theorem 9.1) is an adaptation of the arguments employed in Aubin and Cellina (1984), Theorem 2.4.1, and will be accomplished by constructing a τ_w-Cauchy sequence $\{\Phi_n(t)\}_{n\geq 0}$ of successive approximations of Φ with the property that the sequence $\{(d/dt)\langle\eta, \Phi_n(t)\xi\rangle\}_{n\geq 0}$ is also Cauchy in $\mathbb{C}$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

In what follows, let $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ be arbitrary.

First, observe that by (S_1), $(d/dt)\langle\eta, Y(t)\xi\rangle$ is not necessarily in $P(t, Y(t))(\eta, \xi)$, $t \in I$.

Define Φ_0 to be Y. Then Φ_0 is adapted.

By Aubin and Cellina (1984), Theorem 1.14.2, there is a measurable selection $V_0(\cdot)(\eta, \xi) \in P(\cdot, \Phi_0(\cdot))(\eta, \xi)$ such that

$$|V_0(t)(\eta, \xi) - \frac{d}{dt}\langle\eta, \Phi_0(t)\xi\rangle|$$

$$= \mathbf{d}\left(\frac{d}{dt}\langle\eta, \Phi_0(t)\xi\rangle, P(t, \Phi_0(t))(\eta, \xi)\right) \qquad \text{almost all} \quad t \in J$$

By (S_1), the right-hand side is majorized by $p_{\eta,\xi}(t)$. As the map $(\eta, \xi) \mapsto V_0(t)(\eta, \xi)$ is a sesquilinear form on $\mathbb{D} \underline{\otimes} \mathbb{E}$ for almost all $t \in J$, there is $V_0(t) \in \tilde{\mathscr{A}}$ such that

$$V_0(t)(\eta, \xi) = \langle \eta, V_0(t)\xi \rangle$$

for arbitrary $\eta, \xi \in \mathbb{D} \otimes \mathbb{E}$ and almost all $t \in J$. Since $V_0(\cdot)(\eta, \xi)$ is locally absolutely integrable, $V_0 \in L^1_{\text{loc}}(\tilde{\mathscr{A}})$.

Now define Φ_1 by

$$\Phi_1(t) = x_0 + \int_{t_0}^{t} ds\, V_0(s), \qquad t \in J$$

As $V_0(t) \in \tilde{\mathscr{A}}$ for almost all $t \in J$, it follows that $\Phi_1(t) \in \tilde{\mathscr{A}}_t$, i.e., Φ_1 is adapted. Furthermore, for $t \in J$,

$$\begin{aligned} \|\Phi_1(t) - \Phi_0(t)\|_{\eta,\xi} &\le \|x_0 - \Phi(t_0)\|_{\eta,\xi} \\ &\quad + \int_{t_0}^{t} ds \left| V_0(s)(\eta, \xi) - \frac{d}{ds}\langle \eta, \Phi_0(s)\xi \rangle \right| \\ &\le \delta_{\eta,\xi} + \int_{t_0}^{t} ds\, p_{\eta,\xi}(s) \end{aligned}$$

by (S_1) and (S_4).

Indeed, there exists a sequence $\{\Phi_i\}_{i \ge 0}$ of weakly absolutely continuous maps from I to $\tilde{\mathscr{A}}$ satisfying (i) and (ii) of Proposition (8.1), and hence its assertion.

To prove this claim, assume that $\{\Phi_i\}_{0 \le i \le n}$ has already been defined and satisfies (i) and (ii) of Proposition (8.1). It will be demonstrated that we can find a map $\Phi_{n+1}: J \to \tilde{\mathscr{A}}$ for which (i) and (ii) of Proposition (8.1) also hold.

By Aubin and Cellina (1984), Theorem 1.14.2, there exists $V_n(\cdot)(\eta, \xi) \in P(\cdot, \Phi_n(\cdot))(\eta, \xi)$ such that

$$\begin{aligned} &|\langle \eta, \Phi_n(t)\xi \rangle - V_n(t)(\eta, \xi)| \\ &\quad = \mathbf{d}\left(\frac{d}{dt}\langle \eta, \Phi_n(t)\xi \rangle, P(t, \Phi_n(t))(\eta, \xi)\right), \qquad \text{a.e. on } J \end{aligned}$$

As $(\eta, \xi) \mapsto V_n(t)(\eta, \xi)$ is a sesquilinear form on $\mathbb{D} \otimes \mathbb{E}$ for almost all $t \in J$, there is $V_n \in L^1_{\text{loc}}(\tilde{\mathscr{A}})$ such that $V_n(t)(\eta, \xi) = \langle \eta, \bar{V}_n(t)\xi \rangle$, a.e. on J.

Define Φ_{n+1} by

$$\Phi_{n+1}(t) = x_0 + \int_{t_0}^{t} ds\, V_n(s), \qquad t \in J$$

Then,

$$\begin{aligned} &\left|\frac{d}{dt}\langle \eta, \Phi_{n+1}(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_n(t)\xi\rangle\right| \\ &\quad = |\langle \eta, V_n(t)\xi\rangle - \langle \eta, V_{n-1}(t)\xi\rangle| \\ &\quad \leq P(t, \Phi_n(t))(\eta, \xi), P(t, \Phi_{n-1}(t))(\eta, \xi) \\ &\quad \leq k^P_{\eta,\xi}(t) \|\Phi_n(t) - \Phi_{n-1}(t)\|_{\eta,\xi} \\ &\quad \leq k^P_{\eta,\xi}(t) b_{\eta,\xi,n-1}(t) \end{aligned} \tag{8.1c}$$

which proves (ii)(b) of Proposition 8.1.

Moreover, for $t \in J$,

$$\begin{aligned} &\|\Phi_{n+1}(t) - \Phi_0(t)\|_{\eta,\xi} \\ &\quad \leq \|\Phi_1(t) - \Phi_0(t)\|_{\eta,\xi} + \cdots + \|\Phi_{n+1}(t) - \Phi_n(t)\|_{\eta,\xi} \\ &\quad \leq \sum_{k=0}^{n} b_{\eta,\xi,k}(t) \\ &\quad \leq \Xi_{\eta,\xi}(t) \leq \gamma \end{aligned} \tag{8.1d}$$

This proves (ii)(c) of Proposition 8.1.

It follows from Proposition 8.1(ii)(c) that $\{\Phi_n(t)\}$ is τ_w-Cauchy and converges uniformly to some $\Phi(t)$. Also, from Proposition 8.1(ii)(b), we infer that for almost all $t \in J$, the sequence $\{V_n(t)\}_{n \geq 0}$ is τ_w-Cauchy, whence $\{V_n\}_{n \geq 0}$ converges pointwise a.e. on J to a map $V \in L^1_{loc}(\tilde{\mathscr{A}})$. From Proposition 8.1(ii)(a),

$$\Phi_{i+1}(t) = x_0 + \int_{t_0}^{t} ds\, V_i(s), \qquad i \geq 0$$

Hence,

$$\Phi(t) = x_0 + \int_{t_0}^{t} ds\, V(s), \qquad t \in J$$

As $P(\cdot, \cdot)(\eta, \xi)$ is continuous and has closed values, its graph is closed. Hence, as $\langle \eta, V_i(s)\xi\rangle \in P(s, \Phi_i(s))(\eta, \xi)$ a.e. on J, it follows that

$\langle \eta, V(s)\xi \rangle \in P(s, \Phi(s))(\eta, \xi)$ a.e. on J, whence

$$\frac{d}{ds}\langle \eta, \Phi(s)\xi \rangle \in P(s, \Phi(s))(\eta, \xi), \qquad \text{a.e. on } J$$

Finally, the inequalities (8.1a) and (8.1b) follow from (8.1d) and (8.1c), respectively.

This concludes the proof. ■

Remark. 1. The map Y which features in Theorem 8.2 will be called a *quasisolution* of (4.1), as it is not in general a solution of (4.1).

2. We recall from Notation in Section 3 that $\mathrm{Ad}(\tilde{\mathscr{A}})_{\mathrm{wac}}$ is the set of all *weakly absolutely continuous* adapted stochastic processes.

On $\mathrm{Ad}(\tilde{\mathscr{A}})_{\mathrm{wac}}$ we place the locally convex topology τ^{wac} whose generating family $\{|\cdot|_{\eta,\xi} : \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ of seminorms is defined by

$$|\varphi|_{\eta,\xi} = \|\varphi(t_0)\|_{\eta,\xi} + \int_{t_0}^{T} ds \left| \frac{d}{ds}\langle \eta, \varphi(s)\xi \rangle \right|$$

where I is assumed henceforth to be of the form $I = [t_0, T]$. We write $\mathrm{wac}(\tilde{\mathscr{A}})$ for the completion of the locally convex space $(\mathrm{Ad}(\tilde{\mathscr{A}})_{\mathrm{wac}}, \tau_{\mathrm{wac}})$.

The family $\{|\cdot|_{\eta,\xi} : \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ induces a Hausdorff topology $\tau_{\mathrm{H}}^{\mathrm{wac}}$ on $2^{\mathrm{wac}(\tilde{\mathscr{A}})}$ in a manner similar to that described in Section 2. Denoting the family of pseudometrics which generate this topology by

$$\{\rho_{\eta,\xi}^{\mathrm{wac}} : \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$$

we introduce the notion of a Lipschitzian map from $\tilde{\mathscr{A}}$ into $2^{\mathrm{wac}(\tilde{\mathscr{A}})}$ as in Section 5.

3. Let $\mathfrak{I}$ be the map from $\tilde{\mathscr{A}}$ into $2^{\mathrm{wac}(\tilde{\mathscr{A}})}$ defined by

$$\mathfrak{I}(x) \equiv \{\varphi \in \mathrm{wac}(\tilde{\mathscr{A}}) : \varphi \text{ is a solution of (4.1) satisfying } \varphi(t_0) = x\}$$

Then $\mathfrak{I}$ associates to $x \in \tilde{\mathscr{A}}$ the set of solutions of (4.1) which start from $x \in \tilde{\mathscr{A}}$ at the initial point $t_0 \in I$.

Suppose now that $(x_1, x_2) \in \tilde{\mathscr{A}} \times \tilde{\mathscr{A}}$ with $\|x_1 - x_2\|_{\eta,\xi} \leq \delta_{\eta,\xi} \leq \delta$, $\forall \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Then it follows from Theorem 8.2 that if φ_1 and φ_2 are two solutions of (4.1) such that $\varphi_1(t_0) = x_1$ and $\varphi_2(t_0) = x_2$, we have

$$\begin{aligned}
|\varphi_1 - \varphi_2|_{\eta,\xi} &= \|\varphi_1(t_0) - \varphi_2(t_0)\|_{\eta,\xi} \\
&\quad + \int_{t_0}^{T} ds \left| \frac{d}{ds}\langle \eta, [\varphi_1(s) - \varphi_2(s)]\xi \rangle \right| \\
&\leq \|x_1 - x_2\|_{\eta,\xi} + k_{\eta,\xi}^{P} \int_{t_0}^{T} ds\, \Xi_{\eta,\xi}(s) \\
&\leq \|x_1 - x_2\|_{\eta,\xi} \exp[k_{\eta,\xi}^{P}(T - t_0)]
\end{aligned}$$

$\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. This leads to the following result.

Theorem 8.3. The map $\mathfrak{I}$ from $\tilde{\mathscr{A}}$ into $2^{\mathrm{wac}(\tilde{\mathscr{A}})}$ is Lipschitzian with Lipschitz constants $\{\exp[k^{P}_{\eta,\xi}(T-t_0)]: \eta, \xi\in \mathbb{D}\underline{\otimes}\mathbb{E}\}$. ■

9. A CLOSURE THEOREM

In what follows, T is some fixed positive number and $I=[t_0, T]$. Furthermore, for any $x\in\tilde{\mathscr{A}}$ and $\gamma>0$, introduce

$$Q_{x_0,\gamma}\equiv\{x\in\tilde{\mathscr{A}}: \|x-x_0\|_{\eta,\xi}\leq\gamma \ \forall\eta, \xi\in\mathbb{D}\underline{\otimes}\mathbb{E}\}$$

In this section, E, F, G, and H are maps from $\tilde{\mathscr{A}}$ to clos($\tilde{\mathscr{A}}$) such that $E(X(\cdot))$, $F(X(\cdot))$, $G(X(\cdot))$, and $H(X(\cdot))$ lie in $L^2_{\mathrm{loc}}(\tilde{\mathscr{A}})_{\mathrm{loc}}$ for all $X\in L^2_{\mathrm{loc}}(\tilde{\mathscr{A}})$.

We investigate the relationship between the solutions of the stochastic differential inclusions:

$$\begin{aligned} dX(t)\in E(X(t))\,d\Lambda_\pi(t)+F(X(t))\,dA_f(t)&\\ +G(X(t))\,dA^+_g(t)+H(X(t))\,dt& \end{aligned} \tag{9.1a}$$

almost all $t\in I$,

$$X(t_0)=x_0 \tag{9.1b}$$

and

$$\begin{aligned} dX(t)\in \mathrm{co}\,E(X(t))\,d\Lambda_\pi(t)+\mathrm{co}\,F(X(t))\,dA_f(t)&\\ +\mathrm{co}\,G(X(t))\,dA^+_g(t)+\mathrm{co}\,H(X(t))\,dt& \end{aligned} \tag{9.2a}$$

almost all $t\in I$,

$$X(t_0)=x_0 \tag{9.2b}$$

The following result is a generalization of the *Filippov–Wažewski relaxation theorem* (Aubin and Cellina, 1984, Theorem 2.4.2) (Filippov, 1967, 1971; Wažewski, 1962) for classical differential inclusions.

Theorem 9.1. Let γ be a positive number, x_0 some fixed point in $\tilde{\mathscr{A}}$, and E, F, G, H be Lipschitzian and continuous from $Q_{x_0,\gamma}$ to (comp($\tilde{\mathscr{A}}$), τ_{H}). Then, for every $\varepsilon>0$ and every solution Ψ of (9.2) such that $\Psi(t)$ lies in the interior of $Q_{x_0,\gamma}$ for $t\in I$, there exists a solution Φ of (9.1) such that $\|\Phi(t)-\Psi(t)\|_{\eta,\xi}\leq\varepsilon \ \forall\eta, \xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$, $t\in I$.

Proof. By Theorem 8.2, we need only demonstrate the existence of a *quasisolution* Y of (9.2) on I such that

$$\|\Psi(t)-Y(t)\|_{\eta,\xi}\le\varepsilon/2 \qquad \text{a.e. on } I$$

$$Y(0)=x_0$$

and

$$\mathbf{d}\left(\frac{d}{dt}\langle\eta,\, Y(t)\xi\rangle,\, P(Y(t))(\eta,\,\xi)\right)\le\varepsilon k^{P}_{\eta,\xi}\{2[\exp k^{P}_{\eta,\xi}(T-t_0)-1]\}^{-1}$$

For, by setting $\delta_{\eta,\xi}=0$ and

$$p_{\eta,\xi}=\varepsilon k^{P}_{\eta,\xi}\{2[\exp k^{P}_{\eta,\xi}(T-t_0)-1]\}^{-1}$$

it then follows from Theorem 8.2 that there exists a solution Φ of (9.1) such that

$$\|\Phi(t)-Y(t)\|_{\eta,\xi}\le\varepsilon/2 \qquad \text{on} \qquad I\cap(\text{domain of }\Phi)$$

whence

$$\|\Psi(t)-\Phi(t)\|_{\eta,\xi}\le\varepsilon \qquad \text{a.e. on } I$$

We proceed to construct Y with the alleged properties.

Define

$$I_j\equiv\left[t_0+\frac{(j-1)(T-t_0)}{2n},\, t_0+\frac{j(T-t_0)}{2n}\right]$$

This gives a partition $\{I_j\}_{1\le j\le 2n}$ of I into intervals of length $(T-t_0)/(2n)$.

Denote $\rho(P(Q_{x_0,\gamma})(\eta,\xi),\{0\})$ by $\|P\|_{\eta,\xi}$, $\eta,\xi\in\mathbb{D}\otimes\mathbb{E}$. Then $\|P\|_{\eta,\xi}$ is finite for each $\eta,\xi\in\mathbb{D}\otimes\mathbb{E}$ since $P(Q_{x_0,\gamma})$ is compact ($\bar{P}$ is defined as in Section 6).

By Proposition 5.1, the map co P is Lipschitzian whenever P is such and has the same Lipschitz constants $k^{P}_{\eta,\xi}$ as P. As

$$\rho(P(\Psi(t))(\eta,\xi),\, P(\Psi(s)))(\eta,\xi)\le k^{P}_{\eta,\xi}\|\Psi(t)-\Psi(s)\|_{\eta,\xi}$$

and

$$\langle\eta,[\Psi(t)-\Psi(s)]\xi\rangle\in\int_s^t dr\, P(\Psi(r))(\eta,\xi)$$

whence

$$\|\Psi(t)-\Psi(s)\|_{\eta,\xi}\le\|P\|_{\eta,\xi}\int_{t\wedge s}^{t\vee s}dr=\|P\|_{\eta,\xi}|t-s|$$

it follows that

$$\rho(P(\Psi(t))(\eta, \xi), P(\Psi(s)))(\eta, \xi) \leq \| P\|_{\eta,\xi} k^{P}_{\eta,\xi}|t-s|, \qquad s, t \in I$$

Hence, for $t \in I_j$, the set $(\text{co}\, P)(\Psi(t))(\eta, \xi)$ is contained in an $(\| P\|_{\eta,\xi} k^{P}_{\eta,\xi}(T-t_0)/(2n))$-neighborhood of $(\text{co}\, P)(\Psi(t_j))(\eta, \xi)$. In particular, $(d/dt)\langle \eta, \Psi(t)\xi\rangle$ belongs to this neighborhood for almost all $t \in I_j$ since Ψ is a solution of (9.2).

Let $\{l_{\eta,\xi} : \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ be positive numbers (which will be explicitly specified below) and fix j in $\{1, \ldots, 2n\}$. For $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, let

$$\{S_{k,\eta\xi}\}_{1 \leq k \leq N_j}$$

be a finite Borel partition of $\bigcup \{(\text{co}\, P)(\Psi(t))(\eta, \xi) : t \in I_j\}$ satisfying

$$\sup\{|\lambda_1 - \lambda_2| : \lambda_1, \lambda_2 \in S_{k,\eta\xi}\} \leq l_{\eta,\xi} \qquad \forall k = 1, 2, \ldots, N_j \tag{9.3}$$

and define E_k by

$$E_k \equiv \left\{ t \in \frac{d}{dt}\langle \eta, \Psi(t)\xi\rangle \in S_{k,\eta\xi}, \qquad \forall \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E} \right\}$$

Let $\vartheta_k(\eta, \xi)$ be some point in $S_{k,\eta\xi}$. Then,

$$\mathbf{d}(\vartheta_k(\eta, \xi), (\text{co}\, P)(\Psi(t_j))(\eta, \xi)) \leq \| P\|_{\eta,\xi} k^{P}_{\eta,\xi}(T-t_0)/(2n)$$

and there are finitely many points $Y_{k,l}(\eta, \xi) \in (\text{co}\, P)(\Psi(t_j))(\eta, \xi)$ as well as $\alpha_{k,l} > 0$ satisfying $\sum_l \alpha_{k,l} = 1$ such that

$$\left| \vartheta_k(\eta, \xi) - \sum_l \alpha_{k,l} Y_{k,l}(\eta, \xi) \right| \leq \| P\|^2_{\eta,\xi} k^{P}_{\eta,\xi}/n \tag{9.4}$$

As $(\eta, \xi) \mapsto \vartheta_{\eta,\xi}$ and $(\eta, \xi) \mapsto Y_{k,l}(\eta, \xi)$ are sesquilinear on $\mathbb{D} \underline{\otimes} \mathbb{E}$, there are ϑ_k and $Y_{k,l}$ in $\mathscr{A}$ such that

$$\vartheta_k(\eta, \xi) = \langle \eta, \vartheta_k \xi\rangle$$

and

$$Y_{k,l}(\eta, \xi) = \langle \eta, Y_{k,l}\xi\rangle$$

Then, by (9.3) the simple operator

$$\vartheta(\cdot) = \sum_{k=1}^{N_j} \vartheta_k \chi_{E_k}(\cdot)$$

satisfies

$$\left| \langle \eta, \vartheta_k \xi\rangle - \frac{d}{dt}\langle \eta, \Psi(t)\xi\rangle \right| \leq l_{\eta,\xi} \qquad \text{a.e. on } I_j$$

and

$$\mathbf{d}(\langle \eta, \vartheta_k \xi\rangle, (\text{co}\, P)(\Psi(t_j))(\eta, \xi)) \leq \| P\|_{\eta,\xi} k^{P}_{\eta,\xi}(T-t_0)/(2n)$$

Now the set E_k may be partitioned into measurable subsets E_{kl} such that $|E_{kl}| = \alpha_{kl}|E_k|$, where $|D|$ denotes the Lebesgue measure of D; for how to do this, see Aubin and Cellina (1984), Theorem 2.4.2.

Next, introduce the map $\tilde{Y}: I \to \tilde{\mathscr{A}}$ as the simple map such that $\tilde{Y}|_{E_k} \equiv Y_k$ is given by

$$Y_k(t) = \sum_l Y_{k,l} \chi_{E_{kl}}(t)$$

Then, define $Y: I \to \tilde{\mathscr{A}}$ by

$$Y(t) = x_0 + \int_{t_0}^{t} ds\ \tilde{Y}(s)$$

We shall show that for n sufficiently large, Y is a quasisolution that approximates Ψ. To this end, observe that

$$\frac{d}{dt}\langle \eta, Y(t)\xi \rangle \in P(Q_{x_0,\gamma})(\eta, \xi) \qquad \text{a.e.} \quad \text{on } I$$

showing that Y is Lipschitzian with the same Lipschitz constants $\| P \|_{\eta,\xi}$ as Ψ.

By taking n sufficiently large, it suffices to approximate Ψ at the nodal points $\{t_j\}_{j=1}^{2n}$.

Set

$$c_{\eta,\xi} \equiv \min\{1, (2k_{\eta,\xi}^{P}[\exp k_{\eta,\xi}^{P}(T - t_0) - 1])^{-1}\}$$

Since $t \in I$ implies t lies in some I_j and

$$\| Y(t) - \Psi(t) \|_{\eta,\xi} \le \| Y(t_j) - \Psi(t_j) \|_{\eta,\xi} + \| Y(t) - Y(t_j) \|_{\eta,\xi} + \| \Psi(t) - \Psi(t_j) \|_{\eta,\xi}$$

then as

$$\| \Psi(t) - \Psi(t_j) \|_{\eta,\xi} \le \| P \|_{\eta,\xi} |t - t_j| \le \| P \|_{\eta,\xi}(T - t_0)/(2n)$$

and

$$\| Y(t) - Y(t_j) \|_{\eta,\xi} \le \| P \|_{\eta,\xi}(T - t_0)/(2n)$$

in order to secure the estimate

$$\| Y(t) - \Psi(t) \|_{\eta,\xi} \le c_{\eta,\xi}\varepsilon/3$$

it suffices to have both

$$\| Y(t_j) - \Psi(t) \|_{\eta,\xi} \le c_{\eta,\xi}\varepsilon/6$$

and

$$1/n \le (c_{\eta,\xi}\varepsilon)/[6\| P \|_{\eta,\xi}(T - t_0)] \tag{9.5}$$

This defines a lower bound for n. We establish two other lower bounds for n as follows.

Using

$$\int_{I_j} ds\, \vartheta(s) = \sum_k |E_k| \vartheta_k$$

and

$$\int_{I_j} ds\, \tilde{Y}(s) = \sum_{k,l} \int_{I_j} ds\, Y_{kl} \chi_{E_{kl}}(s) = \sum_{k,l} |E_k| \alpha_{kl} Y_{kl}$$

it follows that

$$\begin{aligned} \left\| \int_{I_j} ds\, [\tilde{Y}(s) - \vartheta(s)] \right\|_{\eta,\xi} &\leq \sum_k |E_k| \left\| \vartheta_k - \sum_l \alpha_{kl} Y_{kl} \right\|_{\eta,\xi} \\ &\leq \sum_k |E_k| (\| P \|_{\eta,\xi}^2 k_{\eta,\xi}^P)/n \qquad \text{[by (9.4)]} \\ &\leq |I_j| (\| P \|_{\eta,\xi}^2 k_{\eta,\xi}^P)/n \end{aligned}$$

If t_r is a point of subdivision, then

$$\begin{aligned} \left\| Y(t_r) - \int_{t_0}^{t_r} ds\, \vartheta(s) \right\|_{\eta,\xi} &= \left\| \int_{t_0}^{t_r} ds\, [\tilde{Y}(s) - \vartheta(s)] \right\|_{\eta,\xi} \\ &= \sum \left\| \int_{I_j} ds\, [\tilde{Y}(s) - \vartheta(s)] \right\|_{\eta,\xi} \\ &\leq [(T - t_0) \| P \|_{\eta,\xi}^2 k_{\eta,\xi}^P]/n \end{aligned}$$

and also

$$\left\| \Psi(t_r) - \int_{t_0}^{t_r} ds\, \vartheta(s) \right\|_{\eta,\xi} \leq l_{\eta,\xi}(T - t_0)$$

Therefore, we may ensure that

$$\| Y(t_r) - \Psi(t_r) \|_{\eta,\xi} \leq c_{\eta,\xi} \varepsilon/6$$

by requiring that

$$(T - t_0) \| P \|_{\eta,\xi}^2 k_{\eta,\xi}^P / n \leq c_{\eta,\xi} \varepsilon/12 \qquad (9.6)$$

thereby obtaining a second lower bound for n, and choosing

$$l_{\eta,\xi} = c_{\eta,\xi} \varepsilon / 12(T - t_0)$$

As $(d/dt)\langle \eta, Y(t)\xi \rangle$ lies in $P(\Psi(t_j))(\eta, \xi)$ whenever the derivative exists, we get

$$\mathbf{d}\left(\frac{d}{dt} \langle \eta, Y(t)\xi \rangle, P(\Psi(t))(\eta, \xi) \right) \leq \| P \|_{\eta,\xi} k_{\eta,\xi}^P (T - t_0)/(2n)$$

and

$$\rho(P(\Psi(t))(\eta, \xi)), P(Y(t))(\eta, \xi)$$
$$\leq k^P_{\eta,\xi}\|\Psi(t) - Y(t)\|_{\eta,\xi} \leq k^P_{\eta,\xi} c_{\eta,\xi}\varepsilon/6 \leq k^P_{\eta,\xi} c_{\eta,\xi}\varepsilon/2$$

Hence

$$\mathbf{d}\left(\frac{d}{dt}\langle\eta, Y(t)\xi\rangle, P(Y(t))(\eta, \xi)\right)$$
$$\leq k^P_{\eta,\xi} c_{\eta,\xi}\varepsilon/2 + (T - t_0)\|P\|_{\eta,\xi} k^P_{\eta,\xi}/2n$$
$$\leq k^P_{\eta,\xi}\varepsilon\{4\exp[k^P_{\eta,\xi}(T - t_0)] - 1\}^{-1} + (T - t_0)\|P\|_{\eta,\xi} k^P_{\eta,\xi}/2n$$

So, by choosing n large enough so as to satisfy (9.4), (9.6), and the constraint

$$(T - t_0)\|P\|_{\eta,\xi}/2n \leq \varepsilon\{4\exp[k^P_{\eta,\xi}(T - t_0)] - 1\}^{-1}$$

we ensure that

$$\|Y(t) - \Psi(t)\|_{\eta,\xi} \leq \varepsilon/2 \qquad \text{for a.e. } t \in I$$

and

$$\mathbf{d}\left(\frac{d}{dt}\langle\eta, Y(t)\xi\rangle, P(Y(t))(\eta, \xi)\right) \leq k^P_{\eta,\xi}\varepsilon\{2\exp[k^P_{\eta,\xi}(T - t_0)] - 1\}^{-1}$$

This concludes the proof. ■

Remark. A member X of $\mathrm{Ad}(\tilde{\mathscr{A}})$ is called *continuous* if the map $t \to X(t)$ of $I = [t_0, T]$ to $\tilde{\mathscr{A}}$ is continuous. Denote the set of all the continuous members of $\mathrm{Ad}(\tilde{\mathscr{A}})$ by $\mathrm{Ad}(\tilde{\mathscr{A}})_{\mathrm{con}}$. We may supply $\mathrm{Ad}(\tilde{\mathscr{A}})_{\mathrm{con}}$ with the locally convex topology τ_{con} whose family $\{\|\cdot\|_{\mathrm{con};\eta,\xi} : \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ of seminorms is defined by

$$\|X\|_{\mathrm{con};\eta,\xi} = \sup_{t \in [t_0, T]} \|X(t)\|_{\eta,\xi}$$

$\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Then, Theorem (9.1) may be rephrased as follows:

Theorem 9.2. The set of solutions to (9.1) is τ_{con}-dense in the set of solutions to (9.2).

Thus, Theorem (9.1) is a *closure* or *density theorem.*

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